

ON SPACES ADMITTING NO ℓ_p OR c_0 SPREADING MODEL

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ABSTRACT. It is shown that for each separable Banach space X not admitting ℓ_1 as a spreading model there is a space Y having X as a quotient and not admitting any ℓ_p for $1 \leq p < \infty$ or c_0 as a spreading model.

We also include the solution to a question of W.B. Johnson and H.P. Rosenthal on the existence of a separable space not admitting as a quotient any space with separable dual.

1. INTRODUCTION

A Banach space X is said to have, for $1 \leq p < \infty$, an ℓ_p spreading model if there is a $\delta > 0$ and a sequence (x_n) in X such that for all $n \in \mathbb{N}$, $n \leq \ell_1 < \dots < \ell_n$ and $(a_i)_i \in c_{00}$,

$$\delta \|(a_i)_{i=1}^n\|_p \leq \left\| \sum_{i=1}^n a_i x_{\ell_i} \right\| \leq \frac{1}{\delta} \|(a_i)_{i=1}^n\|_p.$$

For $p = \infty$ we say X has a c_0 spreading model. The first example of a space not admitting any ℓ_p or c_0 as a spreading model was provided by E. Odell and Th. Schlumprecht in [14]. This space X_S is the completion of $c_{00}(\mathbb{N})$ under a norm that is a modification of the norm of Schlumprecht's space S . As with the norming set of S , the norming set of X_S is defined using the saturation method. In the case of X_S , the norming set includes ℓ_2 convex combination of certain weighted functionals at every step of its, inductive, construction. The idea of including this type of structure in a given norming set can be traced back to work of R.C. James [12] and can also be found in the W.T. Gowers' construction [10] of a space not containing c_0 , ℓ_1 or a reflexive subspace. Recently, in [4], it was shown that there exist hereditarily indecomposable spaces not admitting any ℓ_p or c_0 as a spreading model. In [1], the authors construct a space not admitting an ℓ_p , c_0 or reflexive spreading model. In paper [3] they show that a variant of the space X_S does not admit any ℓ_p or c_0 as a k -iterated spreading model for any $k \in \mathbb{N}$.

In [5] it is shown that every separable Banach space either contains ℓ_1 or is a quotient of a hereditarily indecomposable space. The main theorem of this paper is

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a similar dichotomy for spaces that do not admit ℓ_1 spreading models. By the well-known lifting property of ℓ_1 , if a space X admits an ℓ_1 spreading model, then any Y having X as a quotient must also admit an ℓ_1 spreading model. More precisely, our main theorem is the following dichotomy.

Theorem 1. *Let X be a separable Banach space. Exactly one of the following holds:*

- (1) *X admits an ℓ_1 spreading model.*
- (2) *There is a separable space Y not admitting any ℓ_p for $1 \leq p < \infty$ or c_0 as a spreading model such that X is a quotient of Y .*

We outline the proof of the above theorem: The first step is to pass from a separable space X not admitting an ℓ_1 spreading model to a space Z_X with a bimonotone Schauder basis, having X as a quotient and not having an ℓ_1 spreading model. The second step is to show that for any space Z with a bimonotone Schauder basis and not having an ℓ_1 spreading model, one can construct a *ground set* $G_Z \subset c_{00}$ such that the space Y_{G_Z} , having G_Z as its norming set, also does not have ℓ_1 as a spreading model. After this, using the method in [14], we construct a space $T_{G_Z,2}$ not having any ℓ_p or c_0 as a spreading model. The final, and most difficult, step is to show that the space $T_{G_Z,2}$ has Z as a quotient.

The paper is organized as follows. In section 2 we give several definitions including the definition of a ground set G_Z determined by a space with a basis Z . We also prove the first two steps stated above. In section 3 we define, for a space Z with a basis, the space $T_{G_Z,2}$ and show that it does not admitting any ℓ_p or c_0 spreading model. In section 4 we prove that $T_{G_Z,2}$ has Z as a quotient. We conclude by combining the above to prove our main result and showing that if a space X has as a quotient every space not admitting an ℓ_1 spreading model, then X contains ℓ_1 .

The final section includes a result that is independent from the rest of the paper. Namely, we observe that a space constructed in [2] does not admit as quotient any space with separable dual. This solves a question posed in [13, page 86, Remark IV.1]. We thank W.B. Johnson for bringing this problem to our attention and simplifying our original solution.

2. SPACES HAVING NO ℓ_1 SPREADING MODEL

Let c_{00} be the vector space of all finitely supported scalar sequences and (e_n) denote the unit vector basis of c_{00} . Suppose X has a Schauder basis $(x_n)_{n \in \mathbb{N}}$. Let (x_n^*) be the biorthogonal functionals of (x_n) . For $x \in \text{span}(x_i)_{i=1}^\infty$ let $\text{supp}(x) = \{i : x_i^*(x) \neq 0\}$. Let $B_X = \{x \in X : \|x\| \leq 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$.

Our first definition can be found in [5, Definition 14.1].

Definition 2. *Let Z be a space with a bimonotone Schauder basis $(z_i)_{i \in \mathbb{N}}$ and $(\Lambda_i)_{i \in \mathbb{N}}$ be a partition of \mathbb{N} such that each Λ_i is infinite. Define $G_Z \subset c_{00}$ as*

follows:

$$(1) \quad G_Z = \left\{ \sum_{i=1}^d a_i \left(\sum_{n \in E \cap \Lambda_i} e_n^* \right) : (a_i)_{i=1}^d \subset \mathbb{Q}, \left\| \sum_{i=1}^d a_i z_i^* \right\| \leq 1 \right. \\ \left. E \text{ finite interval of } \mathbb{N} \right\}.$$

G_Z is an example of a ground set. Let Y_{G_Z} be the Banach space that is the completion of c_{00} with the norming set G_Z and (y_n) denote its natural basis.

The space Z is naturally a quotient of Y_{G_Z} . In the next definition, we define the map. In Proposition 6 we will show it is a quotient map.

Definition 3. Define $Q_{G_Z} : Y_{G_Z} \rightarrow Z$ by

$$(2) \quad Q_{G_Z} y_n = e_i \text{ for } n \in \Lambda_i.$$

Notice that for each $i \in \mathbb{N}$ and $(a_j)_{j \in \Lambda_i}$ we have

$$(3) \quad Q_{G_Z} \left(\sum_{j \in \Lambda_i} a_j y_j \right) = \left(\sum_{j \in \Lambda_i} a_j \right) e_i.$$

For an arbitrary separable space X we can construct a space Z_X with a basis that retains many properties of X . The following construction can be found in [5] (also see [16]).

Definition 4. Let X be a separable Banach space. Let $R : \ell_1 \rightarrow X$ be a bounded linear operator such that $(Re_n)_{n=1}^\infty$ is a dense subset of S_X . Let

$$W = \{ER^*x^* : x^* \in B_X \text{ and } E \text{ is an interval of } \mathbb{N}\}.$$

Define the following norm on c_{00} : For $(a_i) \in c_{00}$ let

$$(4) \quad \left\| \sum_i a_i e_i \right\|_{Z_X} = \sup \left\{ f \left(\sum_i a_i e_i \right) : f \in W \right\} \\ = \sup \left\{ \sum_{i \in E} a_i x_i : E \text{ finite interval in } \mathbb{N} \right\}$$

In the above $Re_i = x_i$ for all $i \in \mathbb{N}$. Let Z_X be the completion of c_{00} with the above norm.

Note that Z_X depends on the choice of the dense sequence (x_n) . Note that (e_n) is a bimonotone Schauder basis of Z_X . We now define the natural mapping from Z_X to X . It is easy to see that this map is a bounded quotient map.

Definition 5. Let X be a separable Banach space such that (x_i) is dense in S_X and Z_X be defined as above. Define $Q_X : Z_X \rightarrow X$ by $Q_X(e_i) = x_i$ and extending linearly.

In the next proposition we collect some important facts concerning the spaces and operators defined above. The proofs can be found in [5, Lemmas 14.3 and 14.8].

Proposition 6. *Let X be a separable space and Z be a space with a bimonotone Schauder basis. Then*

- (1) *For every $x \in S_X$ there is a $y \in S_{Z_X}$ such that $Q_X y = x$. In particular, $Q_X : Z_X \rightarrow X$ is a quotient.*
- (2) *If X does not contain ℓ_1 then Z_X does not contain ℓ_1 .*
- (3) *For every $z \in S_Z$ there is a $y \in S_{Y_{G_Z}}$ such that $Q_{G_Z} y = x$. In particular, $Q_{G_Z} : Y_{G_Z} \rightarrow Z$ is a quotient.*
- (4) *If Z does not contain ℓ_1 then Y_{G_Z} does not contain ℓ_1 .*

Proof. We prove only (3). Let $z = \sum_{i=1}^d a_i z_i \in Z$ with $\|z\| = 1$. Let $\ell_i \in \Lambda_i$ for all $i = 1, \dots, d$ and $x = \sum_{i=1}^d a_i e_{\ell_i}$. Clearly $Q_{G_Z} x = z$. We will show that $\|x\| = 1$.

Let $(b_i)_{i=1}^d$ such that $\|\sum_{i=1}^d b_i z_i^*\| \leq 1$ and $\sum_{i=1}^d a_i b_i = (\sum_{i=1}^d b_i z_i^*)(\sum_{i=1}^d a_i z_i) = 1$. By definition $\sum_{i=1}^d b_i e_{\ell_i}^* \in G_Z$. Therefore

$$1 = (\sum_{i=1}^d b_i e_{\ell_i}^*)(\sum_{i=1}^d a_i e_{\ell_i}) \leq \|x\|.$$

Let $\varepsilon > 0$. Find scalars $(c_i)_{i=1}^d$ and an interval E such that

$$\sum_{i \in E} c_i a_i = (\sum_{i \in E} c_i e_{\ell_i}^*)(\sum_{i=1}^d a_i e_{\ell_i}) \geq \frac{\|x\|}{(1 + \varepsilon)}.$$

Using bimonotonicity $\|\sum_{i \in E} c_i z_i^*\| \leq 1$. Therefore

$$\frac{\|x\|}{(1 + \varepsilon)} \leq \sum_{i \in E} c_i a_i \leq (\sum_{i \in E} c_i z_i^*)(\sum_{i=1}^d a_i z_i) \leq 1.$$

Since ε was arbitrary $\|x\| \leq 1$. □

Our next result of this section is the following analogue of Proposition 6 (4).

Proposition 7. *If Z has a basis and does not admit an ℓ_1 spreading model then Y_{G_Z} does not admit an ℓ_1 spreading model.*

Before proving the above, we make a remark that allows us to estimate the norms of vectors in Y_{G_Z} in terms of there images under the quotient map Q_{G_Z} . We also recall an important theorem on the existence of ℓ_1 spreading models in a Banach space not containing ℓ_1 .

Remark 1. Let $\sum_j a_j e_j \in Y_{G_Z}$, then

$$(5) \quad \|\sum_j a_j e_j\|_{G_Z} \leq \sup\{\|Q_{G_Z} P_E(\sum_j a_j e_j)\|_Z : E \text{ is an interval in } \mathbb{N}\}.$$

In the above, $P_E(\sum_j a_j e_j) = \sum_{j \in E} a_j e_j$.

Proof. Let $E \subset \mathbb{N}$ be an interval and $(b_i)_{i=1}^d$ be scalars such that $\|\sum_{i=1}^d b_i z_i^*\| \leq 1$. Using (3)

$$\begin{aligned}
 & |(\sum_{i=1}^d b_i \sum_{j \in E \cap \Lambda_i} e_j^*)(\sum_{i=1}^\infty \sum_{j \in \Lambda_j} a_j e_j)| = |\sum_{i=1}^d b_i (\sum_{j \in \Lambda_i \cap E} a_j)| \\
 & = |(\sum_{i=1}^d b_i e_i^*)(\sum_{i=1}^d (\sum_{j \in \Lambda_j \cap E} a_j) z_i)| \\
 (6) \quad & \leq \|Q_{G_Z} \sum_{i=1}^d (\sum_{j \in \Lambda_j \cap E} a_j z_j)\|_Z \\
 & = \|Q_{G_Z} P_E(\sum_j a_j z_j)\|_Z
 \end{aligned}$$

Since E and $(b_i)_{i=1}^d$ are arbitrary, the remark follows. \square

The next theorem we need due to H.P. Rosenthal [15]. A similar statement can be found in [6].

Theorem 8. *Let X be a Banach space not containing ℓ_1 . Then the following are equivalent.*

- (1) X does not have an ℓ_1 spreading model.
- (2) Every seminormalized weakly null sequence (x_n) has a Cesaro summable subsequence. In other words, there is a subsequence (y_n) of (x_n) such that $\|1/n \sum_{i=1}^n y_i\| \rightarrow 0$ as $n \rightarrow \infty$.

Proof of Proposition 7. Using Proposition 6 (4), Y_{G_Z} does not contain ℓ_1 . Let (z_n) be a seminormalized weakly null sequence in Y_{G_Z} . Our goal is to extract a Cesaro summable subsequence. We pass to a subsequence of (z'_n) of (z_n) that has the following properties:

- (1) (z'_n) is equivalent to a block sequence of (y_n) ;
- (2) $(Q_{G_Z} z'_n)$ is either a bimonotone basic sequence or $\|Q_{G_Z} z'_n\| < 2^{-n}$;
- (3) $(Q_{G_Z} z'_n)$ is Cesaro summable.

Notice that (2) has two cases. Let $\varepsilon > 0$. Find n_0 such that $\|1/n_0 \sum_{i=1}^{n_0} Q_{G_Z} z'_n\| + 3/n_0 < \varepsilon$. Let E be an arbitrary interval. Find n_1, n_2 in \mathbb{N} such that

$$\begin{aligned}
 n_1 &= \min\{n \in \{1, \dots, n_0\} : \text{supp } z'_n \cap E \neq \emptyset\}, \\
 n_2 &= \max\{n \in \{1, \dots, n_0\} : \text{supp } z'_n \cap E \neq \emptyset\}
 \end{aligned}$$

Assume first that $(Q_{G_Z} z'_n)$ is bimonotone basic. Since (z'_n) is a block, for $n_1 < n < n_2$ we have $Q_{G_Z} P_E(z'_n) = Q_{G_Z}(z'_n)$. Using this fact, our assumption on n_0 and fact

that $(Q_{G_Z} z'_n)$ is basic we have

$$(7) \quad \begin{aligned} \|Q_{G_Z} P_E(\frac{1}{n_0} \sum_{n=1}^{n_0} z'_n)\| &= \frac{1}{n_0} \left[\|Q_{G_Z} P_E(z'_{n_1})\| + \left\| \sum_{n=n_1+1}^{n_2-1} Q_{G_Z} z'_n \right\| + \|Q_{G_Z} P_E z'_{n_2}\| \right] \\ &\leq \frac{2}{n_0} + \left\| \frac{1}{n_0} \sum_{i=1}^{n_0} Q z'_n \right\| < \varepsilon. \end{aligned}$$

Since E was arbitrary, applying Remark 1, we finish the proof in the case when $(Q_{G_Z} z'_n)$ bimonotone basic. In the case that $\|Q_{G_Z} z'_n\| < 2^{-n}$ we have

$$(8) \quad \frac{1}{n_0} \left\| \sum_{n=n_1+1}^{n_2-1} Q_{G_Z} z'_n \right\| < \frac{1}{n_0}.$$

Proceeding the same way as in the first inequality of (7), using (8) and the fact that $3/n_0 < \varepsilon$, we finish the proof. \square

The final proposition of this section is analogous to Proposition 6 (2).

Proposition 9. *If X does admit an ℓ_1 spreading model then Z_X does not admit an ℓ_1 spreading model.*

Proof. By Proposition 6 (b) we have that Z_X does not contain ℓ_1 . Therefore, applying Theorem 8 we can consider an arbitrary seminormalized weakly null sequence and show it has a Cesaro summable subsequence. The following remark is a restatement of (4).

Remark 2. Let $\sum_i a_i e_i \in Z_X$ and $P_E(\sum_i a_i e_i) = \sum_{i \in E} a_i e_i$. Then

$$\left\| \sum_i a_i e_i \right\|_Z = \sup \{ \|Q_X P_E(\sum_i a_i e_i)\|_X : E \text{ is an interval in } \mathbb{N} \}.$$

For an arbitrary seminormalized weakly null sequence in Z_X we can pass to a subsequence satisfying the same (1), (2) and (3) as in the proof of Proposition 7. Since Remark 2 is the same as Remark 1 with a different quotient map, by mimicking the proof of Proposition 7 it can be shown that this subsequence in Cesaro summable, as required. \square

3. THE CONSTRUCTION OF $T_{G_Z,2}$ AND SOME PROPERTIES

For the rest of the paper we fix a space Z having a bimonotone Schauder basis and not admitting an ℓ_1 spreading model. In this section we define the space $T_{G_Z,2}$ that does not admit any ℓ_p or c_0 spreading model and has Z as a quotient. To start, we fix two increasing sequences of natural numbers $(m_j)_{j=1}^\infty$ and $(n_j)_{j=1}^\infty$ satisfying:

- (a) $\sum_{i=1}^\infty \frac{1}{m_i} < \frac{1}{10}$.
- (b) $\lim_{i \rightarrow \infty} \frac{((i-1)n_{i-1})^{s_i}}{n_i} = 0$. Where $s_i = \log_{m_1}(m_i)$.
- (c) $\lim_{i \rightarrow \infty} \frac{n_i^\alpha}{m_i} = \infty$ for all $\alpha > 0$.

We now define the norming set inductively. Let $G_0 = G_Z$ (recall the definition from (1)). Suppose G_n has been defined for some $n \geq 0$ define G_{n+1} as follows:

$$G'_{n+1} = \left\{ \frac{1}{m_j} \sum_{i=1}^d f_i : j \in \mathbb{N}, d \leq n_j, (f_i)_{i=1}^d \subset G_n \text{ and } f_1 < \dots < f_d \right\}$$

$$G''_{n+1} = \left\{ \sum_{i=1}^n \lambda_i f_i : n \in \mathbb{N}, \lambda_i \geq 0, \sum_{i=1}^n \lambda_i^2 \leq 1, (f_i)_{i=1}^n \subset G'_{n+1}, w(f_i) = m_i \right\}$$

Let $G_{n+1} = G''_{n+1} \cup G_n$. Let $D_{G_Z} = \bigcup_{n=1}^{\infty} G_n$.

Let $T_{G_Z,2}$ be the completion of c_{00} under the norm $\|x\|_{D_{G_Z}} = \{f(x) : f \in D_G\}$.

Notation 1. Let $f \in G_n \setminus G_Z$ for some $n \in \mathbb{N}$.

- (1) If $f \in G'_n$ then $f = 1/m_j \sum_{i=1}^d f_i$ for some $j \in \mathbb{N}$. In this case we say f is weighted and set the ‘weight of f ’ $w(f) = m_j$. Note that this weight is not unique.
- (2) If $f \in G''_n$ then $f = \sum_{i=1}^k \lambda_i f_i$ where $w(f_i) = m_i$ and $\sum_{i=1}^k \lambda_i^2 \leq 1$. Set $w(f) = \{m_i : \lambda_i \neq 0\}$. If $|w(f)| > 1$ we say f is not weighted.
- (3) For $\sum_{i=1}^k \lambda_i f_i \in G''_n$ let $f_{\leq i_0} = \sum_{i=1}^{i_0} \lambda_i f_i$ and $f_{> i_0} = \sum_{i=i_0+1}^k \lambda_i f_i$.

A variant of the next theorem can be found in [4, Theorem 11.3]. We include the proof here to give a more complete presentation.

Theorem 10. *Let Z be a space with a bimonotone Schauder basis not having an ℓ_1 spreading model. Then $T_{G_Z,2}$ does not have any ℓ_p or c_0 as a spreading model.*

Before passing to the proof we state two lemmas.

Lemma 11. *Suppose $y \in c_{00}$ and $\varepsilon > 0$. There is an $i_0 \in \mathbb{N}$ such that for all $f \in D_{G_Z}$, $f_{> i_0}(y) < \varepsilon$.*

Proof. Let i_0 such that $\sum_{i > i_0} |\text{supp } y|/m_i < \varepsilon$. The evaluation follows easily. \square

The next lemma follows from standard arguments which, in the interest of brevity, we omit.

Lemma 12. *Let $f \in D_{G_Z} \setminus G_Z$ such that $w(f) = \{m_{j_0}\}$ for some $j_0 \in \mathbb{N}$. Let $j > j_0$ and $(x_i)_{i=1}^{n_j}$ be a normalized block sequence in $T_{G_Z,2}$. Then*

$$f\left(\frac{1}{n_j} \sum_{i=1}^{n_j} x_i\right) < \frac{3}{m_{j_0}}.$$

Proof of Theorem 10. It is easy to see that neither ℓ_p , for $1 < p < \infty$, nor c_0 is finitely block representable in $T_{G_Z,2}$ and therefore can not be admitted as a spreading model. Indeed, let $(y_k)_{k=1}^{\infty}$ be a block sequence in $T_{G_Z,2}$. For every $i \in \mathbb{N}$ we have

$$\left\| \sum_{k=1}^{n_i} y_k \right\| \geq \frac{n_i}{m_i}.$$

We have assumed that for all $\alpha > 0$, $\lim_{i \rightarrow \infty} n_i^\alpha / m_i = \infty$ (assumption (c)). Therefore for no $p > 1$ does there exist a C_p such that for every $i \in \mathbb{N}$, $\|\sum_{k=1}^{n_i} y_k\| \leq C_p n_i^{\frac{1}{p}}$.

It remains to show that $T_{G_Z, 2}$ does not admit an ℓ_1 spreading model. Let (w_n) be a bounded sequence generating an ℓ_1 spreading model. We must pass to further subsequences of (w_n) to achieve additional properties. First, it is well-known that since $(w_n) \subset X$ generates an ℓ_1 -spreading model then for $0 < \varepsilon < 10^{-4}$ we can find a block sequence (y_n) of (w_n) which generates a $(1 - \varepsilon)$ - ℓ_1 spreading model. Secondly, since Y_{G_Z} does not admit an ℓ_1 spreading model, we may apply the Erdos-Madigor theorem [8] to find an $n_0 \in \mathbb{N}$ and a block sequence (z_n) of (y_n) such that $z_n = \sum_{i \in F_n} x_i / n_0$ where $|F_n| = n_0$ for all $n \in \mathbb{N}$ and $\|z_n\|_{G_Z} < \varepsilon$. Passing to a further subsequence of $(z_n)_n$ (for example, $(z_{kn_0})_{k=1}^\infty$) we have a subsequence (x_n) of (z_n) satisfying

- (x_n) generates an ℓ_1 spreading model with constant $(1 - \varepsilon)$.
- $\|x_n\|_{G_Z} < \varepsilon$ for all $n \in \mathbb{N}$.

The next step is to prove the following claim.

Claim 13. *There is an $i_0 \in \mathbb{N}$ such that for each $n > 2$ there is a $\psi^n \in D_{G_Z} \setminus G_Z$ satisfying*

- (a) $w(\psi^n) \leq m_{i_0}$;
- (b) $\psi^n(x_n) > 1 - 4\sqrt{\varepsilon}$.

Since (x_n) is a $(1 - \varepsilon)$ - ℓ_1 spreading model for each $n > 2$ there is a $\phi^n = \sum_{i=1}^k \lambda_i^n \phi_i^n$ such that $\phi^n(x_2 + x_n) > 2(1 - \varepsilon)$. It follows that

$$\phi^n(x_2) > 1 - 2\varepsilon \text{ and } \phi^n(x_n) > 1 - 2\varepsilon.$$

Apply Lemma 11 for x_2 and ε to find an i_0 such that for each $n \geq 2$, $\phi_{>i_0}^n(x_2) < \varepsilon$. We claim that $\phi_{\leq i_0}^n$ is our desired ψ^n . By definition $\phi_{\leq i_0}^n$ satisfies (a). It suffices to prove that (b) holds. Notice that

$$(9) \quad \phi_{\leq i_0}^n(x_2) = \phi^n(x_2) - \phi_{>i_0}^n(x_2) > 1 - 3\varepsilon.$$

Now observe that

$$(10) \quad \phi_{\leq i_0}^n(x_n) > 1 - 2\varepsilon - \sum_{i=i_0+1}^k \lambda_i^n \phi_i^n(x_n).$$

Using (9)

$$(11) \quad \left(\sum_{i=1}^{i_0} (\lambda_i^n)^2 \right)^{\frac{1}{2}} \geq \phi_{\leq i_0}^n(x_2) > 1 - 3\varepsilon.$$

From (11) we have,

$$(12) \quad \left(\sum_{i=i_0+1}^{\infty} \lambda_i^n \phi_i^n(x_n) \right)^2 \leq \sum_{i=i_0+1}^{\infty} (\lambda_i^n)^2 = \sum_{i=1}^{\infty} (\lambda_i^n)^2 - \sum_{i=1}^{i_0} (\lambda_i^n)^2 < 3\varepsilon.$$

Combining (10), (12) and the fact that $2\varepsilon + \sqrt{3\varepsilon} < 4\sqrt{\varepsilon}$, (b) follows.

What (a) and (b) together tell us is that for *every* $n > 2$ there is a functional ψ^n which almost norms x_n and has only ‘small’ (less than some fixed m_{i_0}) weights. This allows use to show, in the the next lemma, that no element in the sequence $(x_n)_n$ can be normed by functionals with weights larger than m_{i_0} .

Lemma 14. *Let $n > 2$ and $\phi \in D_{G_Z}$ with $w(\phi) > m_{i_0}$. Then*

$$\phi(x_n) < \frac{1}{2}.$$

Proof. Let $\phi \in D_{G_Z}$ with $w(\phi) > m_{i_0}$. Then $f = (\psi^n + \phi)/\sqrt{2} \in D_{G_Z}$. Using Claim 13 (b)

$$\phi(x_n) = \sqrt{2}f(x_n) - \psi^n(x_n) < \sqrt{2} - 1 + (4\sqrt{\varepsilon}) < \frac{1}{2}.$$

As desired. \square

We can now arrive at a contradiction using the following vector

$$z = \frac{1}{n_{i_0+1}} \sum_{q=1}^{n_{i_0+1}} x_{n_{i_0+1}+q}.$$

Find $\phi \in D_{G_Z}$ such that $\phi(z) > 1 - \varepsilon$. Using the Lemma 12 and Lemma 14 we have:

$$(13) \quad .9 < 1 - \varepsilon < \phi(z) = \phi_{\leq i_0}(z) + \phi_{> i_0}(z) < \sum_{i=1}^{i_0} \lambda_i \frac{3}{m_i} + \frac{1}{2} < \frac{3}{10} + \frac{1}{2}.$$

This is a contradiction. \square

We now describe the tree decomposition of the functionals in D_{G_Z} . First we must set some notation. Let $\mathbb{N}^{<\mathbb{N}}$ be the set of all finite tuples of \mathbb{N} . For $\delta, \gamma \in \mathbb{N}^{<\mathbb{N}}$ we write $\delta \prec \gamma$ if δ is an initial segment of γ . Let $\gamma(i)$ the the i^{th} coordinate of γ . Let \mathbb{N}^d denote the set of d -tuples of \mathbb{N} and $\mathbb{N}^{\leq d} = \cup_{i \leq d} \mathbb{N}^i$. For $\gamma \in \mathbb{N}^d$ let $Im_\gamma \subset \mathbb{N}^{d+1}$ denote the immediate successors of γ . The following proposition describes a decomposition of the functionals in D_{G_Z} . Tree decompositions are a ubiquitous component in constructions of this type. As such, we omit the proof of the proposition.

Proposition 15. *Let $n \in \mathbb{N}$ and $f \in G_n \setminus G_0$. Then there is a set $\mathcal{T}_f \subset \mathbb{N}^{\leq 2n} \cup \{\emptyset\}$ and a collection $(f_\gamma)_{\gamma \in \mathcal{T}_f}$ of functionals which we call a tree decomposition satisfying the following properties:*

- (1) $f_\emptyset = f$.
- (2) Let $S_\gamma^f = Im_\gamma \cap \mathcal{T}_f$ and $\mathcal{T}_f^d = \mathcal{T}_f \cap \mathbb{N}^d$. If $\gamma \in \mathcal{T}_f$ and $S_\gamma^f = \emptyset$ we say that γ is a terminal node. In this case, $f_\gamma \in G_0$.

(3) Let $0 \leq k < n$. If $\gamma \in \mathcal{T}_f^{(2k)}$ then

$$f_\gamma = \sum_{\delta \in S_\gamma^f} \lambda_\delta f_\delta \text{ where } w(f_\delta) = m_{\delta(2k+1)}, \sum_{\delta \in S_\gamma^f} \lambda_\delta^2 \leq 1$$

If $\gamma \in \mathcal{T}_f^{(2k+1)}$, then

$$f_\gamma = \frac{1}{m_{\gamma(2k+1)}} \sum_{\delta \in S_\gamma^f} f_\delta.$$

where $(f_\delta)_{\delta \in S_\gamma^f}$ are successive and $|S_\gamma^f| \leq n_{\gamma(2k+1)}$.

We need one more definition.

Definition 16. Let $f \in D_{G_Z} \setminus G_Z$ and $\mathcal{T}_f \subset \mathbb{N}^{<\mathbb{N}} \cup \{\emptyset\}$ such that the collection $(f_\gamma)_{\gamma \in \mathcal{T}_f}$ is a tree decomposition.

- (1) For $\alpha \in \mathcal{T}_f$ let $|\alpha| = k$ whenever $\alpha \in \mathbb{N}^k$.
- (2) Let $\mathcal{M}_f = \{\alpha \in \mathcal{T}_f : \alpha \text{ is a terminal node of } \mathcal{T}_f\}$.

4. Z IS A QUOTIENT OF $T_{G_Z, 2}$

As the title above suggests, the main objective of this section is to prove that Z is a quotient of $T_{G_Z, 2}$. After we establish this, we will prove the main theorem and one proposition. To begin we require two lemmas.

Lemma 17. Let $n \in \mathbb{N}$ and $f \in D_{G_Z}$. Suppose that for all $\alpha \in \mathcal{M}_f$, $|\alpha| \geq 2n$. Then $\|f\|_\infty \leq 10^{-n}$.

Proof. We proceed by induction on n . For $n = 1$ we have

$$\|f\|_\infty = \left\| \sum_{\delta \in S_\emptyset} \frac{\lambda_\delta}{m_{\delta(1)}} \sum_{\beta \in S_\delta} f_\beta \right\|_\infty \leq \sum_{\delta \in S_\emptyset} \frac{1}{m_{\delta(1)}} \sup_{\beta \in S_\delta} \|f_\beta\|_\infty < \frac{1}{10}.$$

In the above we used the for each $\delta \in S_\emptyset$, the functionals $(f_\beta)_{\beta \in S_\delta}$ have disjoint support. Assume the claim for some $n \geq 1$. We will prove it for $n + 1$.

$$\|f\|_\infty = \left\| \sum_{\delta \in S_\emptyset} \frac{\lambda_\delta}{m_{\delta(1)}} \sum_{\beta \in S_\delta} f_\beta \right\|_\infty \leq \sum_{\delta \in S_\emptyset} \frac{1}{m_{\delta(1)}} \sup_{\beta \in S_\delta} \|f_\beta\|_\infty < \sum_{j=1}^{\infty} \frac{1}{m_j} \frac{1}{10^n} \leq \frac{1}{10^{n+1}}$$

In the above we used that for each $\delta \in S_\emptyset$, the functionals $(f_\beta)_{\beta \in S_\delta}$ have disjoint support and have terminal nodes each of height greater than $2n$. \square

Lemma 18. Let $j_0 \in \mathbb{N}$ and $f \in D_{G_Z}$ such that for all $\alpha \in \mathcal{M}_f$ there is a $\beta \prec \alpha$ such that f_β is weighted and $w(f_\beta) \geq m_{j_0}$. Then $\|f\|_\infty \leq 2 \sum_{j \geq j_0} \frac{1}{m_j}$.

Proof. For every $\alpha \in \mathcal{M}_f$ let

$$\beta_\alpha = \min\{\beta : \beta \prec \alpha, f_\beta \text{ is weighted and } w(f_\beta) \geq m_{j_0}\}.$$

Notice that if $\alpha \neq \alpha'$ are in \mathcal{M}_f then β_α is either equal to or not comparable with $\beta_{\alpha'}$. We will prove the following by induction: For all $\gamma \in \mathcal{T}_f$ such that there is an $\alpha \in \mathcal{M}_f$ with $\gamma \preceq \beta_\alpha$ one of the following holds:

- (1) If $\gamma = \beta_\alpha$ for some $\alpha \in \mathcal{M}_f$ then $\|f_\gamma\|_\infty \leq \frac{1}{w(f_\gamma)}$.
- (2) If $\gamma \prec \beta_\alpha$ for all $\alpha \in \mathcal{M}_f$ with $\gamma \prec \alpha$ and f_γ is weighted then

$$\|f_\gamma\|_\infty \leq \frac{2}{w(f_\gamma)} \sum_{j \geq j_0} \frac{1}{m_j}.$$

- (3) If $\gamma \prec \beta_\alpha$ for all $\alpha \in \mathcal{M}_f$ with $\gamma \prec \alpha$ and f_γ is not weighted then

$$\|f_\gamma\|_\infty \leq 2 \sum_{j \geq j_0} \frac{1}{m_j}.$$

After we prove the above, by taking $\gamma = \emptyset$, the lemma follows.

For the base case of the induction, we suppose that $\gamma = \beta_\alpha$ for some $\alpha \in \mathcal{M}_f$. Since it is clear that for all $\alpha \in \mathcal{M}_f$, $\|f_{\beta_\alpha}\|_\infty \leq 1/w(f_{\beta_\alpha})$, we are done.

Let $\gamma \in \mathcal{T}_f$ such that $\gamma \prec \beta_\alpha$ for all $\alpha \in \mathcal{M}_f$ with $\gamma \prec \alpha$. Assume that for all $\tilde{\gamma}$ with $\gamma \prec \tilde{\gamma} \preceq \beta_\alpha$ for some β_α , (1), (2) or (3) holds (depending on $\tilde{\gamma}$).

Assume that f_γ weighted. Then

$$(14) \quad \|f_\gamma\|_\infty = \left\| \frac{1}{w(f_\gamma)} \sum_{\delta \in S_\gamma} f_\delta \right\|_\infty \leq \frac{1}{w(f_\gamma)} \max_{\delta \in S_\gamma} \|f_\delta\|_\infty \leq \frac{2}{w(f_\gamma)} \sum_{j \geq j_0} \frac{1}{m_j}$$

In the above we used the induction hypothesis for $\delta \in S_\gamma$ since $\gamma \prec \delta \preceq \beta_\alpha$ whenever $\gamma \prec \beta_\alpha$. Note that if $\delta = \beta_\alpha$ then $\|f_\delta\|_\infty \leq 1/m_{j_0} < 2 \sum_{j \geq j_0} 1/m_j$.

Assume that f_γ is not weighted. Let $A_\gamma = \{\delta \in S_\gamma : \delta = \beta_\alpha, \alpha \in \mathcal{M}_f\}$. Splitting the set S_γ and applying the induction hypothesis we have

$$(15) \quad \begin{aligned} \|f_\gamma\|_\infty &\leq \sum_{\delta \in S_\gamma} \|f_\delta\|_\infty = \sum_{\delta \in A_\gamma} \|f_\delta\|_\infty + \sum_{\delta \in S_\gamma \setminus A_\gamma} \|f_\delta\|_\infty \\ &\leq \sum_{j \geq j_0} \frac{1}{m_j} + \sum_{\delta \in S_\gamma \setminus A_\gamma} \frac{1}{w(f_\delta)} \sum_{j \geq j_0} \frac{1}{m_j} \leq 2 \sum_{j \geq j_0} \frac{1}{m_j}. \end{aligned}$$

In the above we used that $\sum_{\delta \in S_\gamma \setminus A_\gamma} \frac{1}{w(f_\delta)} < 1$. □

We are now ready to prove the main proposition of this section.

Proposition 19. *Let $Q : T_{G_Z, 2} \rightarrow Z$ be the bounded linear map defined by $Q(e_i) = z_n$ for $i \in \Lambda_n$. Then Q is a quotient map.*

Notice that Q makes the same identifications as the map Q_{G_Z} from Definition 3.

Proof. Let $z = \sum_{i=1}^d a_i z_i \in Z$ such that $\|z\|_Z = 1$. We will construct a vector x such that $Qx = z$ and $\|x\| = 1$; of course, this is sufficient to prove the proposition. Assume that $j_0 \in \mathbb{N}$ satisfies the following:

- (1) $\sum_{j \geq j_0} \frac{2d}{m_j} < \frac{1}{5}$
- (2) $\frac{2((j_0-1)n_{j_0-1})^{s_{j_0}}}{n_{j_0}} < \frac{1}{5d}$
- (3) $\frac{d}{10^{s_{j_0}}} < \frac{1}{5}$

For each $t \in \{1, \dots, n_j\}$ let $(\ell_t^i)_{i=1}^d \subset \Lambda_i$ such that

$$\ell_t^1 < \ell_t^2 < \dots < \ell_t^d < \ell_{t+1}^1 < \dots.$$

Now set

$$x = \frac{1}{n_{j_0}} \sum_{t=1}^{n_{j_0}} \sum_{i=1}^d a_i e_{\ell_t^i} = \sum_{i=1}^d a_i \sum_{t=1}^{n_{j_0}} \frac{1}{n_{j_0}} e_{\ell_t^i}.$$

Let $y_t = \sum_i a_i e_{\ell_t^i}$. Note that $(y_t)_{t=1}^{n_{j_0}}$ is a block sequence and $Qx = z$. It is easy to see for all $t \in \{1, \dots, n_{j_0}\}$,

$$(16) \quad \|y_t\| \leq \|y_t\|_1 \leq \sum_{i=1}^d |a_i| \leq d.$$

and

$$(17) \quad \|x\| \leq \|x\|_1 \leq \sum_{i=1}^d |a_i| \leq d.$$

We will also need the following easy remark

Remark 3. Let $g \in G_Z$ and $t \in \{1, \dots, n_{j_0}\}$. Then $g(y_t) \leq 1$.

Note that for all $t \in \{1, \dots, n_{j_0}\}$, $Q_{G_Z}(y_t) = z$. Since $\|z\| = 1$ we can apply Proposition 6 (3) to deduce that $\|y_t\|_{G_Z} = 1$. The remark follows.

We observe first that $\|x\| \geq 1$: Suppose $(b_i)_{i=1}^d$ is a scalar sequence such that

$$\sum_{i=1}^d a_i b_i = \left(\sum_{i=1}^d b_i z_i^* \right) \left(\sum_{i=1}^d a_i z_i \right) = 1.$$

By definition $\sum_{i=1}^d b_i \left(\sum_{t=1}^{n_{j_0}} e_{\ell_t^i}^* \right) \in G_Z$. Thus

$$\|x\| \leq \left(\sum_{i=1}^d b_i \sum_{t=1}^{n_{j_0}} e_{\ell_t^i}^* \right) \left(\sum_{i=1}^d a_i \frac{1}{n_{j_0}} \sum_{t=1}^{n_{j_0}} e_{\ell_t^i} \right) = \sum_{i=1}^d a_i b_i = 1.$$

Therefore, for $f \in D_G$ it suffices to show that $f(x) \leq 1$. Partition \mathcal{M}_f as follows:

$$\begin{aligned} A_1 &= \{\alpha \in \mathcal{M}_f : |\alpha| \geq 2s_{j_0}\} \\ A_2 &= \{\alpha \in \mathcal{M}_f : \exists \beta \prec \alpha, w(f_\beta) \geq m_{j_0}\} \\ A_3 &= \mathcal{M}_f \setminus (A_1 \cup A_2) \end{aligned}$$

Let $f = f_1 + f_2 + f_3$ such that for $i \in \{1, 2, 3\}$, f_i has A_i as its terminal nodes. This splits the rest of the proof naturally into three separate cases. The first two cases are taken care of by Lemmas 17 and 18 respectively.

Using Lemma 17, (17) and condition (3) on j_0

$$(18) \quad |f_1(x)| \leq \|f_1\|_\infty \|x\|_1 \leq \frac{d}{10^{s_{j_0}}} < \frac{1}{5}.$$

Similarly, using Lemma 18, (17) and condition (1) on j_0 we have

$$(19) \quad |f_2(x)| \leq \|f_2\|_\infty \|x\|_1 \leq 2d \sum_{j \geq j_0} \frac{1}{m_j} < \frac{1}{5}.$$

To estimate $|f_3(x)|$ it is convenient to separate the support of x into 2 sets. Let

$$(20) \quad E_2 = \{t \in \{1, \dots, n_{j_0}\} : \exists \alpha \in A_3, \text{supp} y_t \cap \text{range} g_\alpha \neq \emptyset \text{ and } \text{supp} y_t \not\subset \text{range} g_\alpha\}$$

and $E_1 = \{1, \dots, n_{j_0}\} \setminus E_2$.

First we bound $|E_2|$ (the cardinality of E_2). Observe that $|E_2| < 2|A_3|$. Indeed, for each $t \in E_2$ there is an $\alpha \in A_3$ and each $\alpha \in A_3$ corresponds to at most 2 elements of E_2 . By definition, for $\alpha \in A_3$: $|\alpha| < 2s_{j_0}$ and for all $\beta \prec \alpha$ such that that f_β is weighted, $w(f_\beta) \leq m_{j_0-1}$. These facts together yield that $|A_3| \leq ((j_0 - 1)n_{j_0-1})^{s_{j_0}}$.

Using the above along with condition (2) on j_0 we conclude that

$$(21) \quad |E_2| < 2|A_3| \leq 2((j_0 - 1)n_{j_0-1})^{s_{j_0}} < n_{j_0}/(5d)$$

Using (21) and (16) we have

$$(22) \quad f_3\left(\frac{1}{n_{j_0}} \sum_{t \in E_2} y_t\right) \leq \frac{1}{n_{j_0}} \sum_{t \in E_2} \|y_t\| \leq \frac{d|E_2|}{n_{j_0}} < \frac{1}{5}$$

We now pass to the final evaluation. Let $x_1 = \sum_{t \in E_1} y_t$. For $\gamma \in \mathcal{T}_{f_3}$ let

$$(23) \quad I_\gamma = \{t \in \{1, \dots, n_{j_0}\} : \text{supp} y_t \subset \text{range} f_\gamma\}.$$

Let $\gamma \in \mathcal{T}_{f_3}$, we will prove the following:

- (1) If $\gamma \in A_3$ then $|f_\gamma(x_1)| \leq |I_\gamma|$.
- (2) If $\gamma \notin A_3$ and f_γ is weighted then

$$|f_\gamma(x_1)| \leq \frac{2}{w(f_\beta)} |I_\gamma|.$$

- (3) If $\gamma \notin A_3$ and f_γ is not weighted then

$$|f_\gamma(x_1)| \leq \frac{1}{5} |I_\gamma|.$$

The proof goes by induction (and is similar to the proof of Lemma 18). For the base case we assume that $\gamma \in A_3$. Using Remark 3 we have

$$|f_\gamma(x_1)| \leq |f_\gamma(\sum_{t \in I_\gamma} y_t)| \leq |I_\gamma|.$$

Assume that $\gamma \notin A_3$ and that for all γ' with $\gamma \prec \gamma'$ either (1), (2) or (3) holds (depending on γ').

Our first case is when f_γ is weighted. Splitting the sum and applying the appropriate induction hypothesis we have

$$\begin{aligned}
 |f_\gamma(x_1)| &\leq \frac{1}{w(f_\gamma)} \sum_{\delta \in S_\gamma \cap A_3} |f_\delta(x_1)| + \frac{1}{w(f_\gamma)} \sum_{\delta \in S_\gamma \setminus A_3} |f_\delta(x_1)| \\
 (24) \quad &\leq \frac{1}{w(f_\gamma)} \left(\sum_{\delta \in S_\gamma \cap A_3} |I_\delta| + \frac{1}{5} \sum_{\delta \in S_\gamma \setminus A_3} |I_\delta| \right) \\
 &\leq \frac{2}{w(f_\gamma)} |I_\gamma|
 \end{aligned}$$

Assuming γ is not weighted, we again apply the induction hypothesis to get the desired estimate.

$$\begin{aligned}
 |f_\gamma(x_1)| &\leq \sum_{\delta \in S_\gamma} |f_\delta(x_1)| \leq \sum_{\delta \in S_\gamma} \frac{2}{w(f_\delta)} |I_\delta| \\
 (25) \quad &\leq 2 \max_{\delta \in S_\gamma} |I_\delta| \sum_{j=1}^{\infty} \frac{1}{m_j} \leq \frac{1}{5} |I_\gamma|.
 \end{aligned}$$

The inductive proof is finished. It follows that

$$(26) \quad |f_3(\frac{1}{n_j} \sum_{t \in E_1} y_t)| = \frac{1}{n_j} |f_3(x_1)| \leq \frac{|E_1|}{5n_j} \leq \frac{1}{5}.$$

Combining (18), (19), (22) and (26) we have

$$|f(x)| \leq |f_1(x)| + |f_2(x)| + |f_3(x)| < \frac{4}{5} < 1.$$

This finishes the proof of the proposition. \square

We can now prove our main theorem. Of course, all that is required is to apply our previous work and compose quotient maps.

Proof of Theorem 1. Let X be a separable Banach space not admitting an ℓ_1 spreading model. By Proposition 9 the space Z_X has a basis and does not admit an ℓ_1 spreading model. Moreover the map $Q_X : Z_X \rightarrow X$ is a quotient map. Let $Z_X = Z$. Define G_Z as in (1) and $T_{G_Z,2}$ as above. Theorem 10 says that $T_{G_Z,2}$ has no ℓ_p or c_0 spreading model. Theorem 19 yields that the map $Q : T_{G_Z,2} \rightarrow Z$ is a quotient. $Q_X \circ Q : T_{G_Z,2} \rightarrow X$ is the desired quotient. \square

We conclude with one last proposition that relates to our main theorem. In particular, we note that there does not exist a space Y not admitting any ℓ_p or c_0 as a spreading model and having, as a quotient, *every* space X not admitting an ℓ_1 spreading model. In other words, there is no *universal* space satisfying the requirements of our theorem.

Proposition 20. *Suppose X has as a quotient every space not admitting an ℓ_1 spreading model. Then X contains a copy of ℓ_1 .*

Proof. Recall that if the Bourgain ℓ_1 -index [7] of a space is unbounded (i.e. equals ω_1) then the space contains ℓ_1 . The main result of [4] states that for each countable ordinal ξ there is a separable space X_ξ that does not admit an ℓ_1 spreading model and has hereditary ℓ_1 -index greater than ω^ξ . If a space X has, as quotient, every space not admitting an ℓ_1 spreading model it must have the space X_ξ as quotient for each $\xi < \omega_1$. It follows that X must have unbounded Bourgain index.

Looking more closely at the construction of X_ξ , one can observe that the *ground space* X_{G_ξ} on which X_ξ is built also does not admit an ℓ_1 spreading model and has ℓ_1 -index greater than ω^ξ (just not hereditarily).

Finally, we give the reader a concrete example: Consider the following unconditional James tree space: Let $J_{2,1}$ to be the completion of $c_{00}(\mathbb{N}^{<\mathbb{N}})$ equipped with the norm

$$(27) \quad \|z\| = \sup \left\{ \left(\sum_{i=1}^d \left(\sum_{t \in \mathfrak{s}_i} |z(t)| \right)^2 \right)^{1/2} \right\}$$

where the above supremum is taken over all families $(\mathfrak{s}_i)_{i=1}^d$ of pairwise incomparable non-empty segments of $\mathbb{N}^{<\mathbb{N}}$. For every well-founded tree S of natural numbers, let $J_{2,1}^S$ be the closed subspace supported on the coordinates of S . Using arguments similar to those in [4], for every well-founded tree S , the space $J_{2,1}^S$ has no ℓ_1 spreading model. It is easy to see that the Bourgain ℓ_1 index of $J_{2,1}^S$ is at least the height of the well founded tree S . Arguing as before, we conclude that any space having each $J_{2,1}^S$ as a quotient must contain ℓ_1 . \square

5. SPACES NOT ADMITTING QUOTIENTS WITH SEPARABLE DUALS

In this section we answer affirmatively a problem posed in [13, Remark VI]. The problem asks if there exists a separable Banach space X such that every infinite dimensional quotient has a non separable dual. We note that the dual of such a space is closely connected to HI spaces. Indeed, the dual X^* must be non separable and cannot contain c_0 , ℓ_1 or a reflexive subspace. Therefore, it does not contain a subspace with an unconditional basis [12]. W. T. Gowers' dichotomy [11] yields that X^* is saturated with HI spaces which do not contain a reflexive subspaces. Next, we provide some sufficient conditions for the existence of a space answering the Johnson-Rosenthal question in the affirmative. We note that the sufficient conditions in the following theorem are quite close to being necessary.

Theorem 21. *Let X be a Banach space with the following properties:*

- (1) *X does not contain a reflexive subspace.*
- (2) *X^* is separable.*
- (3) *X^{**} is hereditarily indecomposable.*

Then the dual Y^ of any quotient Y of X^* is non-separable.*

Proof. Assume on the contrary that there exists a quotient Y of X^* with Y^* separable. As it is shown in [13], Y has a further quotient with a shrinking basis. Therefore, we assume that Y has a shrinking basis $(y_n)_{n \in \mathbb{N}}$ and that the biorthogonal functionals $(y_n^*)_{n \in \mathbb{N}}$ form a boundedly complete basis of Y^* , which is isomorphic to a subspace of X^{**} . It follows that there exists a normalized boundedly complete basic sequence $(w_n^{**})_{n \in \mathbb{N}}$ in X^{**} . We will show that this yields a contradiction. Indeed, since X^{**} is HI, there exists a normalized sequence $(z_n)_{n \in \mathbb{N}}$ in X that is equivalent to a block sequence of $(w_n^{**})_{n \in \mathbb{N}}$; hence, $(z_n)_{n \in \mathbb{N}}$ is also boundedly complete. Since X^* is separable, the sequence $(z_n)_{n \in \mathbb{N}}$ has a further block sequence $(v_n)_{n \in \mathbb{N}}$ which is normalized and shrinking [13]. The sequence $(v_n)_{n \in \mathbb{N}}$ remains boundedly complete and hence $Z = \overline{\langle (v_k)_{k \in \mathbb{N}} \rangle}$ is reflexive. This contradicts assumption (i). \square

Corollary 22. *There exists a separable Banach space X such that every infinite dimensional quotient has non separable dual.*

Proof. In [2] a Banach space Z is constructed satisfying the assumptions of Theorem 21. Z^* is the desired space. \square

To conclude, we state the following problem that was communicated to the authors by W.B. Johnson.

Question 1. Does every separable space have a quotient which is either HI or has an unconditional basis?

This problem is a natural analogue of Gowers' dichotomy for quotients. In relation to this problem, V. Ferenczi [9] proved a dichotomy for quotients of subspaces of Banach spaces. In particular, we recommend section 3 of this paper which contains several interesting questions and observations relating to these types of problems.

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